7-2 The Quadratic Formula

Objective To solve quadratic equations by using the quadratic formula.

Quadratic equations are used in many applications, so it is useful to have a formula that gives their solutions directly from the coefficients. You can derive this formula by applying the method of completing the square to the general quadratic equation:

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \text{ (Recall that } a \neq 0.)$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{Complete the square.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2} \qquad \text{Factor the left side.}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1 Solve $3x^2 + x - 1 = 0$.

Solution For the equation $3x^2 + 1x - 1 = 0$,

$$a = 3,\ b = 1, \text{ and } c = -1.$$

Substitute these values in the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{-1 \pm \sqrt{1 - (-12)}}{6} = \frac{-1 \pm \sqrt{13}}{6}$$

$\therefore$ the solution set is $\left\{\frac{-1 + \sqrt{13}}{6}, \frac{-1 - \sqrt{13}}{6}\right\}$. ***Answer***

The solutions just obtained are exact and expressed in simplest radical form. In applications you may want approximate solutions to the nearest hundredth. Since $\sqrt{13} \approx 3.606$ (from a calculator or Table 1, page 810),

$$x \approx \frac{-1 + 3.606}{6} \quad \text{or} \quad x \approx \frac{-1 - 3.606}{6}$$

$$x \approx 0.43 \quad \text{or} \quad x \approx -0.77$$

$\therefore$ the solution set is $\{0.43, -0.77\}$. ***Answer***

Example 2 Solve $5y^2 = 6y - 3$.

Solution First rewrite the equation in the form $ax^2 + bx + c = 0$:

$$5y^2 - 6y + 3 = 0$$

Then substitute 5 for a, -6 for b, and 3 for c in the quadratic formula.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(3)}}{2(5)}$$

$$= \frac{6 \pm \sqrt{-24}}{10}$$

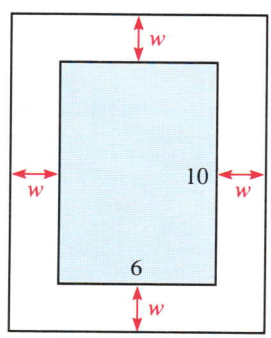
$$= \frac{6 \pm 2i\sqrt{6}}{10}$$

$$= \frac{3 \pm i\sqrt{6}}{5}$$

$\therefore$ the solution set is $\left\{\frac{3 + i\sqrt{6}}{5}, \frac{3 - i\sqrt{6}}{5}\right\}$. ***Answer***

In Examples 1 and 2, the coefficients a, b, and c were integers. Keep in mind, however, that the quadratic formula can be used to solve *any* quadratic equation, whether the coefficients are fractions, decimals, irrational numbers, or imaginary numbers. (See Exercises 29–40, page 314.)

Example 3 A swimming pool 6 m wide and 10 m long is to be surrounded by a walk of uniform width. The area of the walk happens to equal the area of the pool. What is the width of the walk?



Solution

1. Make a sketch.
   Let w = the width of the walk.
   Then the dimensions of the pool plus the walkway are $10 + 2w$ by $6 + 2w$.

2. The area of the pool is $10 \cdot 6 = 60$ (m^2).
The area of the walk can be determined by subtracting the pool's area from the total area covered by both the pool and the walk.

$$\begin{aligned} \text{Area of walk} &= \text{Total area} - \text{Area of pool} \\ &= (10 + 2w)(6 + 2w) - 60 \end{aligned}$$

3.
$$\begin{aligned} \text{Area of walk} &= \text{Area of pool} \\ (10 + 2w)(6 + 2w) - 60 &= 60 \\ (60 + 32w + 4w^2) - 60 &= 60 \\ 4w^2 + 32w &= 60 \\ 4w^2 + 32w - 60 &= 0 \longleftarrow \text{Divide both sides by 4.} \\ w^2 + 8w - 15 &= 0 \end{aligned}$$

4. By completing the square or using the quadratic formula you will find that

$$w = -4 \pm \sqrt{31}.$$

Since $-4 - \sqrt{31} \approx -9.57$, you must reject this root because the width of the walk cannot be negative.

$\therefore$ the width of the walk is $(-4 + \sqrt{31})$ m, or approximately 1.57 m. ***Answer***

5. *Check:* A calculator is helpful in checking approximate solutions.

$$\begin{aligned} \text{Area of walk} &= \text{Area of pool} \\ (10 + 2w)(6 + 2w) - 60 &\overset{?}{\approx} 60 \\ (10 + 2 \cdot 1.57)(6 + 2 \cdot 1.57) - 60 &\overset{?}{\approx} 60 \\ (13.14)(9.14) - 60 &\overset{?}{\approx} 60 \\ 120.10 - 60 &\overset{?}{\approx} 60 \quad \checkmark \end{aligned}$$

Oral Exercises

Give the values that you would substitute for *a*, *b*, and *c* in the quadratic formula.

1. $2x^2 - 3x + 7 = 0$
2. $3x^2 + 7x - 2 = 0$
3. $5 - 7x - 4x^2 = 0$
4. $x^2 = 4 - 2x$
5. $x^2 - x\sqrt{5} + 1 = 0$
6. $x(x - 2) = 9$

Written Exercises

Solve each equation. Give answers involving radicals in simplest radical form.

A

1. $x^2 + 6x + 4 = 0$
2. $v^2 + 3v - 5 = 0$
3. $y^2 - 4y + 13 = 0$
4. $t^2 + 6t + 6 = 0$
5. $5k^2 + 3k - 2 = 0$
6. $2p^2 - 3p - 2 = 0$

Solve each equation after rewriting it in the form $ax^2 + bx + c = 0$. Give answers involving radicals in simplest radical form.

7. $5r^2 + 8 = -12r$
8. $2w^2 + 4w = -3$
9. $3y^2 = 1 - y$
10. $8x = 1 - x^2$
11. $2x(x + 1) = 7$
12. $5 = 4r(2r + 3)$
13. $(3n - 5)(2n - 2) = 6$
14. $(2x + 1)(2x - 1) = 4x$
15. $\frac{w^2}{2} - w = \frac{3}{4}$
16. $\frac{t^2}{2} + 1 = \frac{t}{5}$
17. $\frac{2m^2 + 16}{5} = 2m$
18. $\frac{4 - 2y^2}{7} = 2y$

Solve each equation and approximate solutions to the nearest hundredth. A calculator may be helpful.

19. $2n^2 - 4n = 8$
20. $2x^2 - 3x = 7$
21. $3t^2 - 6t - 7 = 0$
22. $4x(x + 1) = 2.75$
23. $3x(x + 2) = -2.5$
24. $2t(t - 4) = -3$

Solve each equation (a) by factoring and (b) by using the quadratic formula.

25. $5x^2 - 45 = 0$
26. $3y^2 - 48 = 0$
27. $3x^2 - 6x + 3 = 0$
28. $4y^2 + 4y - 15 = 0$

Solve each equation. Give answers involving radicals in simplest radical form.

B 29. $x^2 - x\sqrt{2} - 1 = 0$
30. $x^2 - x\sqrt{5} - 1 = 0$
31. $t^2 - 2t\sqrt{2} + 1 = 0$
32. $u^2 + 2u\sqrt{3} - 3 = 0$
33. $\sqrt{2}x^2 + 5x + 2\sqrt{2} = 0$
34. $\sqrt{3}x^2 - 2x + 2\sqrt{3} = 0$
35. $z^2 + iz + 2 = 0$
36. $z^2 + 2iz - 1 = 0$
37. $z^2 - (3 + 2i)z + (1 + 3i) = 0$
38. $iz^2 + (2 - 3i)z - (3 + i) = 0$
39. $\frac{2w + i}{w - i} = \frac{3w + 4i}{w + 3i}$
40. $\frac{1}{2z + i} + \frac{1}{2z - i} = \frac{4}{z + 2i}$
41. Show that the solutions of $3x^2 - 2x + 3 = 0$ are reciprocals.

C 42. Prove that if the roots of $ax^2 + bx + c = 0$ $(a \neq 0)$ are reciprocals, then $a = c$.

Problems

Solve each problem. Approximate any answers involving radicals to the nearest hundredth. A calculator may be helpful.

A 1. Each side of a square is 4 m long. When each side is increased by x m, the area is doubled. Find the value of x.

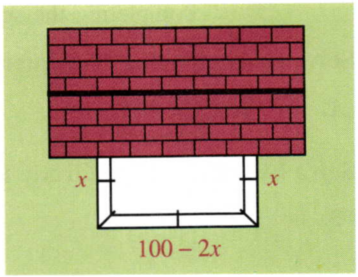
2. A rectangle is 6 cm long and 5 cm wide. When each dimension is increased by x cm, the area is tripled. Find the value of x.

3. A positive real number is 1 more than its reciprocal. Find the number.

4. Two positive real numbers have a sum of 5 and product of 5. Find the numbers.

5. A rectangular field with area 5000 m^2 is enclosed by 300 m of fencing. Find the dimensions of the field.

6. A rectangular animal pen with area 1200 m^2 has one side along a barn. The other three sides are enclosed by 100 m of fencing. Find the dimensions of the pen.

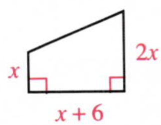


7. A walkway of uniform width has area 72 m^2 and surrounds a swimming pool that is 8 m wide and 10 m long. Find the width of the walkway.

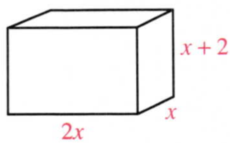
8. A 5 in. by 7 in. photograph is surrounded by a frame of uniform width. The area of the frame equals the area of the photograph. Find the width of the frame.

9. When mineral deposits formed a coating 1 mm thick on the inside of a pipe, the area through which fluid can flow was reduced by 20%. Find the original inside diameter of the pipe.
(*Remember:* Area of circle = πr^2 and diameter = $2r$.)

10. The area of the trapezoid shown below is 90 square units. Find the value of x.



Ex. 10



Ex. 11

B 11. The total surface area of the rectangular solid shown is 36 m^2. Find the value of x.

12. In a *golden rectangle* the ratio of the length to the width equals the ratio of the length plus width to the length. Find the value of this *golden ratio*. (Do not approximate the answer.)

13. A box with height $(x + 5)$ cm has a square base with side x cm. A second box with height $(x + 2)$ cm has a square base with side $(x + 1)$ cm. If the two boxes have the same volume, find the value of x.

14. A box with a square base and no lid is to be made from a square piece of metal by cutting squares from the corners and folding up the sides. The cut-off squares are 5 cm on a side. If the volume of the box is 100 cm^3, find the dimensions of the original piece of metal.

15. A hydrofoil made a round trip of 144 km in 4 h. Because of head winds, the average speed on returning was 15 km/h less than the average speed going out. Find the two speeds.

Self-Test 1

Vocabulary quadratic equation (p. 307) quadratic formula (p. 311)
completing the square (p. 308)

Solve by completing the square.

1. $x^2 - 6x + 2 = 0$ **2.** $3y^2 + 9y = -2$ **Obj. 7-1, p. 307**

Solve by using the quadratic formula.

3. $w^2 - 5w = 3$ **4.** $4z^2 + 2z + 1 = 0$ **Obj. 7-2, p. 311**

Solve each problem. Approximate answers involving radicals to the nearest hundredth. A calculator may be helpful.

5. Find the dimensions of a rectangle whose perimeter is 10 cm and whose area is 3 cm^2.

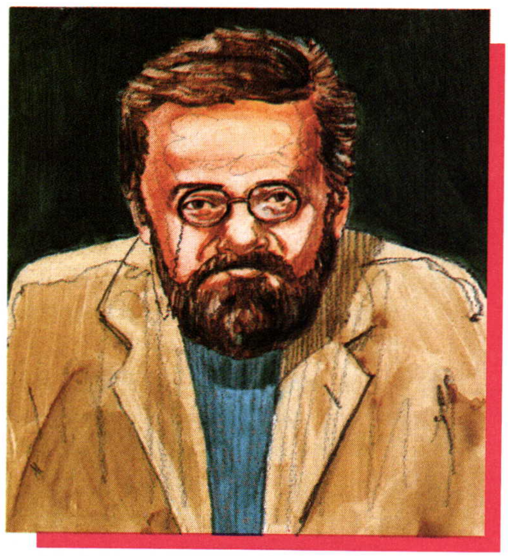
6. A sidewalk of uniform width has area 180 ft^2 and surrounds a flower bed that is 11 ft wide and 13 ft long. Find the width of the sidewalk.

Check your answers with those at the back of the book.

Biographical Note / *Charles Steinmetz*

Charles Proteus Steinmetz (1865–1923) was an electrical engineer and mathematician who combined technical skill with theoretical insight. He showed that sophisticated mathematics could help to solve problems in the design of motors and transformers, and he pioneered the use of complex numbers in the analysis of alternating current circuits.

Born in Germany, Steinmetz immigrated to the United States at the age of 24 and was soon working for an electric company, where he spent most of his career. Even though he suffered from a deformed spine, Steinmetz had a tremendous capacity for work. In his laboratory he devised and improved designs for arc-lamp electrodes and generators and studied the effects of transient currents like those produced by lightning. Steinmetz also researched solar energy, electrical networks, electrification of railways,



synthetic production of protein, and electric cars. His 195 patents are proof of the range of his invention.